

# On the isolated singularities of the solutions of the Gaussian curvature equation for nonnegative curvature

Daniela Kraus and Oliver Roth

Universität Würzburg, Mathematisches Institut,  
D-97074 Würzburg, Germany  
dakraus@mathematik.uni-wuerzburg.de  
roth@mathematik.uni-wuerzburg.de

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**Abstract.** The precise asymptotic behaviour of the solutions to the twodimensional curvature equation  $\Delta u = k(z) e^{2u}$  with  $e^{2u} \in L^1$  for bounded nonnegative curvature functions  $-k(z)$  near isolated singularities is obtained.

## 1 Results

The aim of this note is to classify the isolated singularities of the real-valued solutions with finite energy of the twodimensional curvature equation  $\Delta u = k(z) e^{2u}$  in the case of *variable* nonnegative curvature. We first consider solutions  $u \in C^2(\mathbb{D} \setminus \{0\})$  of this equation, where  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  denotes the open unit disk in the complex plane.

### Theorem 1.1

Let  $k \in C(\mathbb{D} \setminus \{0\})$  be a bounded and nonpositive function and  $u \in C^2(\mathbb{D} \setminus \{0\})$  a real-valued solution to the curvature equation

$$\Delta u = k(z) e^{2u} \quad (1.1)$$

with

$$\iint_{\mathbb{D}} e^{2u(z)} dx dy < \infty. \quad (1.2)$$

Then there exists a real number  $\gamma > -1$  such that

$$u(z) = \gamma \log |z| + r(z) \quad (1.3)$$

where  $r \in C(\mathbb{D})$ . In addition,  $r \in C^1(\mathbb{D})$  if  $\gamma > -1/2$ , and  $r \in C^2(\mathbb{D})$  if  $\gamma \geq 0$  and if  $k$  is locally Hölder continuous on  $\mathbb{D}$ .

### Remark 1.2

- (a) For  $k(z) \equiv -4$  the PDE (1.1) reduces to the *Liouville equation*  $\Delta u = -4 e^{2u}$ . In this case Theorem 1.1 was proved by Chou and Wan [4, 5] using complex analysis and Liouville's classical representation formula [12] for the solutions to  $\Delta u = -4 e^{2u}$ . In the variable curvature case some (nonsharp) estimates for the solutions  $u$  of (1.1) satisfying the energy estimate (1.2) have recently been obtained by Yunyan [24] using blow-up analysis and the moving plane method.

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- (b) If  $k(z)$  is *strictly* positive (and bounded), then condition (1.2) is redundant. The asymptotic behaviour of the solutions in this case was found by McOwen [14] (see also Heins [8]); the regularity properties of the remainder function are studied in [11]. We note that the isolated singularities of the solutions in the classical constant case  $k(z) = +4$  were first described by J. Nitsche [16] and later by Chou and Wan [4, 5].
- (c) The “energy condition” (1.2) cannot be dispensed with. This can already be seen in the constant case  $k(z) = -4$ . Here one obtains very badly behaved solutions

$$u(z) = \log \left( \frac{|g'(z)|}{1 + |g(z)|^2} \right)$$

at  $z = 0$  by choosing  $g$  meromorphic on  $\mathbb{D} \setminus \{0\}$  with an essential singularity at  $z = 0$  (see [4]). Condition (1.2) seems to be a “canonical” assumption when dealing with nonnegative curvature (see e.g. Brézis & Merle [3], Hang & Wang [7], Jost, Wang & Zhou [10], Yunyan [24] and the references cited therein).

- (d) The punctured unit disk plays no special role in Theorem 1.1– we can replace it by any domain in the complex plane with an isolated boundary point and obtain a corresponding version of Theorem 1.1 in this more general situation. This will become apparent from the proof of Theorem 1.1.

Geometrically, every function  $u$  of Theorem 1.1 gives rise to a conformal Riemannian metric  $\lambda(z) |dz| := e^{u(z)} |dz|$  with a *conical singularity of order  $\gamma > -1$*  at  $z = 0$ , i.e.,  $\lambda(z) |dz| = |z|^\gamma e^{r(z)} |dz|$  and curvature  $-k(z)$ . Theorem 1.1 contains precise information how the connection (sometimes also called Pre-Schwarzian or Christoffel symbol)

$$\Gamma_\lambda(z) := 2 \frac{\partial \log \lambda(z)}{\partial z}$$

of  $\lambda(z) |dz|$  and its projective connection (see [10]) or Schwarzian (see [15])

$$S_\lambda(z) := \frac{\partial \Gamma_\lambda(z)}{\partial z} - \frac{1}{2} \Gamma_\lambda(z)^2 = 2 \left[ \frac{\partial^2 \log \lambda(z)}{\partial z^2} - \left( \frac{\partial \log \lambda(z)}{\partial z} \right)^2 \right]$$

behave at the conical singularity:

### Corollary 1.3

Let  $\lambda(z) |dz|$  be a conformal metric on  $\mathbb{D} \setminus \{0\}$  with curvature  $-k(z)$  for some nonpositive bounded continuous function  $k : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{R}$  and  $\iint_{\mathbb{D}} \lambda(z)^2 dx dy < \infty$ . Then

- (a)  $\lim_{z \rightarrow 0} z \Gamma_\lambda(z) = \gamma$ ; and
- (b)  $\lim_{z \rightarrow 0} z^2 S_\lambda(z) = -\gamma(2 + \gamma)/2$  if  $k$  is locally Hölder continuous on  $\mathbb{D}$ ,

where  $\gamma > -1$  is the order of the conical singularity of  $\lambda(z) |dz|$  at  $z = 0$ .

### Remark 1.4

The Schwarzian  $S_\lambda$  of a conformal metric plays an important role in particular for metrics with *constant* curvature. This classical constant curvature case is intimately related to complex analysis. In fact, if  $\lambda(z) |dz|$  has constant curvature, then  $S_\lambda$  is a *holomorphic* function with isolated singularities exactly at the isolated singularities of the metric  $\lambda(z) |dz|$ . Corollary

1.3 shows that for constantly curved singular metrics with  $\iint \lambda(z)^2 dx dy < \infty$  the Schwarzian  $S_\lambda$  has a pole of order 2 at the conical singularities. We refer to [1, 2, 4, 6, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 22, 23] for more information about conformal metrics with constant curvature and conical singularities.

## 2 Proofs

Theorem 1.1 follows from the following lemma, the Brézis–Merle lemma [3] and standard elliptic regularity results.

### Lemma 2.1

Let  $k \in C(\mathbb{D} \setminus \{0\})$  be a bounded nonpositive continuous function and let  $u \in C^2(\mathbb{D} \setminus \{0\})$  be a real-valued solution to the curvature equation (1.1) which satisfies (1.2). Then there exists a real number  $\gamma > -1$  and a harmonic function  $h$  on  $\mathbb{D}$  such that

$$u(z) = \gamma \log |z| + h(z) + v(z),$$

where

$$v(z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \log |z - \zeta| k(\zeta) e^{2u(\zeta)} d\sigma_\zeta$$

is the Newton potential of  $k(z) e^{2u(z)}$  and  $d\sigma_\zeta$  denotes two-dimensional Lebesgue measure w.r.t.  $\zeta$ .

### Proof.

(a) It turns out that it is more appropriate to work with the *nonpositive* Green potential of  $k(z) e^{2u(z)}$  instead of its Newton potential  $v$ . We therefore let

$$g_{\mathbb{D}}(z, \zeta) := -\log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right|, \quad z, \zeta \in \mathbb{D}$$

denote Green's function on  $\mathbb{D}$ , i.e.,  $g(z, \zeta) \geq 0$ , and let

$$q(z) := \frac{1}{2\pi} \iint_{\mathbb{D}} g_{\mathbb{D}}(z, \zeta) k(\zeta) e^{2u(\zeta)} d\sigma_\zeta.$$

Thus  $q$  is a well-defined nonpositive continuous function on  $\mathbb{D} \setminus \{0\}$  and can be decomposed as  $q(z) = -v(z) + h_1(z)$ , where  $v$  is the Newton potential of  $k(z) e^{2u(z)}$  and  $h_1$  is a harmonic function in  $\mathbb{D}$  (including  $z = 0$ ). Thus  $q$  is subharmonic in  $\mathbb{D}$  (including  $z = 0$ ; we do not know yet whether  $q(0) > -\infty$ ), see [20, Theorem 3.1.2.]. Now, let

$$w := q + u.$$

Then  $w$  is continuous on  $\mathbb{D} \setminus \{0\}$  and

$$\Delta w(z) = \Delta q(z) + \Delta u(z) = -k(z) e^{2u(z)} + k(z) e^{2u(z)} = 0 \quad \text{in } \mathbb{D} \setminus \{0\}$$

in the sense of distributions, see [20, p. 74]. By Weyl's lemma,  $w$  is harmonic in  $\mathbb{D} \setminus \{0\}$ . Therefore

$$w(z) = \gamma \log |z| + \operatorname{Re} g(z),$$

for some constant  $\gamma \in \mathbb{R}$  and some holomorphic function  $g : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{C}$ .

(b) We now prove that  $g$  is in fact holomorphic on the whole unit disk  $\mathbb{D}$ . If not, then  $g$  would have a nonremovable singularity at  $z = 0$ , so  $e^g$  would have an essential singularity at  $z = 0$ :

$$e^{g(z)} = \sum_{n=-\infty}^{\infty} b_n z^n, \quad b_n \neq 0 \text{ for infinitely many } n < 0. \quad (2.1)$$

However, using Parseval's identity,

$$\begin{aligned} 2\pi \sum_{n=-\infty}^{\infty} |b_n|^2 \int_0^1 r^{2n+2\gamma+1} dr &= \int_0^1 \int_0^{2\pi} \left| \sum_{n=-\infty}^{\infty} b_n r^n e^{int} \right|^2 dt r^{2\gamma+1} dr = \iint_{\mathbb{D}} |z|^{2\gamma} |e^{g(z)}|^2 dx dy \\ &\leq \iint_{\mathbb{D}} e^{2u(z)} dx dy < \infty. \end{aligned}$$

Consequently,  $b_n = 0$  for all  $n < -1 - \gamma$ , which contradicts (2.1).

(c) We next show that  $\gamma > -1$ .

Since  $g$  is holomorphic in  $\mathbb{D}$ , we have

$$u(z) = w(z) - q(z) = \gamma \log |z| + \operatorname{Re}(g(z)) - q(z) \geq \gamma \log |z| + \operatorname{Re}(g(z)) \geq \gamma \log |z| + c$$

in  $|z| < 1/2$  for some real constant  $c$ , so

$$e^{2u(z)} \geq |z|^{2\gamma} |e^{2g(z)}| \geq |z|^{2\gamma} e^{2c}, \quad |z| < 1/2.$$

Thus the assumption (1.2) implies

$$\iint_{|z| < 1/2} |z|^{2\gamma} dx dy < \infty,$$

so  $\gamma > -1$ .

(d) Returning to what we have proved in (a), we see that  $u(z) = \gamma \log |z| + h(z) + v(z)$ , where  $h(z) := \operatorname{Re}(g(z)) - h_1(z)$  is harmonic in  $\mathbb{D}$  and  $v$  is the Newton potential of  $k(z) e^{2u(z)}$ . ■

**Lemma 2.2 (Brézis–Merle [3])**

Let  $f \in L^1(\mathbb{D})$  and let  $v(z)$  be the Newton potential of  $f$ . Then

$$e^{|v|} \in L^p(\mathbb{D}) \quad \text{for every } 0 < p < +\infty.$$

**Proof of Theorem 1.1.** By Lemma 2.1,  $u(z) = \gamma \log |z| + r(z)$ , where  $r \in C^2(\mathbb{D} \setminus \{0\})$ . Thus, we only need to investigate the regularity properties of the remainder function  $r(z)$  in *some* neighborhood of  $z = 0$ , say in  $\mathbb{D}_{1/2} = \{z \in \mathbb{C} : |z| < 1/2\}$ . Let  $v$  be the Newton potential of  $f(z) := k(z) e^{2u(z)}$ . By assumption  $f \in L^1(\mathbb{D}) \cap C(\mathbb{D} \setminus \{0\})$ , so Lemma 2.2 shows  $e^v \in L^p(\mathbb{D})$  for any  $0 < p < +\infty$ . Using Lemma 2.1 we can write  $u(z) = \gamma \log |z| + r(z)$  for some  $\gamma > -1$  with  $r = h + v$  where  $h$  is harmonic in  $\mathbb{D}$ . We thus get

$$f(z) = k(z) |z|^{2\gamma} e^{2h(z)} e^{2v(z)}.$$

We first consider the case  $-1 < \gamma < 0$ . Then  $f \in L^q(\mathbb{D}_{1/2})$  for every  $1 < q < -1/\gamma$ . The regularity properties of the Newton potential show that  $v$  belongs to the Sobolev space

$W^{2,q}(\mathbb{D}_{1/2})$  for any  $1 < q < -1/\gamma$ , see [9, Theorem 9.2.1]. Thus the Sobolev embedding theorems yield  $r = h + v \in C(\mathbb{D}_{1/2})$  if  $-1 < \gamma \leq -1/2$  and  $r = h + v \in C^1(\mathbb{D}_{1/2})$  if  $-1/2 < \gamma < 0$ . The case  $\gamma \geq 0$  is easier, because now  $f \in L^q(\mathbb{D}_{1/2})$  for any  $1 < q < \infty$ . Hence  $v \in W^{2,q}(\mathbb{D}_{1/2})$  for any  $1 < q < \infty$ , so  $r = h + v \in C^1(\mathbb{D}_{1/2})$ . If  $k$  is locally Hölder continuous on  $\mathbb{D}$  (and  $\gamma \geq 0$ ), then  $f$  is locally Hölder continuous on  $\mathbb{D}$ , so  $r = h + v \in C^2(\mathbb{D})$ . ■

**Proof of Corollary 1.3.** We first consider the case  $\gamma \geq 0$ . Then  $u(z) = \log \lambda(z) = \gamma \log |z| + r(z)$  with  $r \in C^1(\mathbb{D})$  by Theorem 1.1, so

$$z \Gamma_\lambda(z) = \gamma + 2z \frac{\partial r}{\partial z}(z) \rightarrow \gamma \quad \text{as } z \rightarrow 0.$$

If  $k$  is locally Hölder continuous in  $\mathbb{D}$ , then  $r \in C^2(\mathbb{D})$  and

$$z^2 S_\lambda(z) = -\frac{\gamma(2+\gamma)}{2} - 2\gamma z \frac{\partial r}{\partial z}(z) + 2z^2 \left( \frac{\partial^2 r}{\partial z^2}(z) - \left( \frac{\partial r}{\partial z}(z) \right)^2 \right) \xrightarrow{z \rightarrow 0} -\frac{\gamma(2+\gamma)}{2}.$$

Now let  $-1 < \gamma < 0$ . Choose a positive integer  $m$  with  $\gamma^* := \gamma m + m - 1 \geq 0$ . Let  $\pi_m : \mathbb{D} \setminus \{0\} \rightarrow \mathbb{D} \setminus \{0\}$  be the holomorphic cover projection  $\pi_m(z) = z^m$ . Then the pullback  $\lambda^*(z) |dz| := m|z|^{m-1} \lambda(z^m) |dz|$  of  $\lambda(z) |dz|$  via  $\pi_m$  induces a conformal metric on  $\mathbb{D} \setminus \{0\}$  with curvature  $-k(z^m)$  and  $u^*(z) := \log \lambda^*(z) = \gamma^* \log |z| + r^*(z)$ . Moreover, using the transformation rules for the connection and the projective connection (see [15, p. 334]),

$$\Gamma_{\lambda^*}(z) = m \Gamma_\lambda(z^m) z^{m-1} + \frac{m-1}{z}, \quad S_{\lambda^*}(z) = S_\lambda(z^m) m^2 z^{2(m-1)} - \frac{m^2-1}{2z^2},$$

we get

$$\lim_{z \rightarrow 0} z \Gamma_\lambda(z) = \lim_{z \rightarrow 0} z^m \Gamma_\lambda(z^m) = \frac{\gamma^* - (m-1)}{m} = \gamma$$

and, if  $k$  is locally Hölder continuous on  $\mathbb{D}$ , then

$$\lim_{z \rightarrow 0} z^2 S_\lambda(z) = \lim_{z \rightarrow 0} z^{2m} S_\lambda(z^m) = \frac{-\gamma^*(2+\gamma^*) + m^2 - 1}{2m^2} = -\frac{\gamma(2+\gamma)}{2}.$$

■

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